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A VARIATIONAL INEQUALITY APPROACH TO THE BELLMAN-DIRICHLET EQUA--ETC(U)

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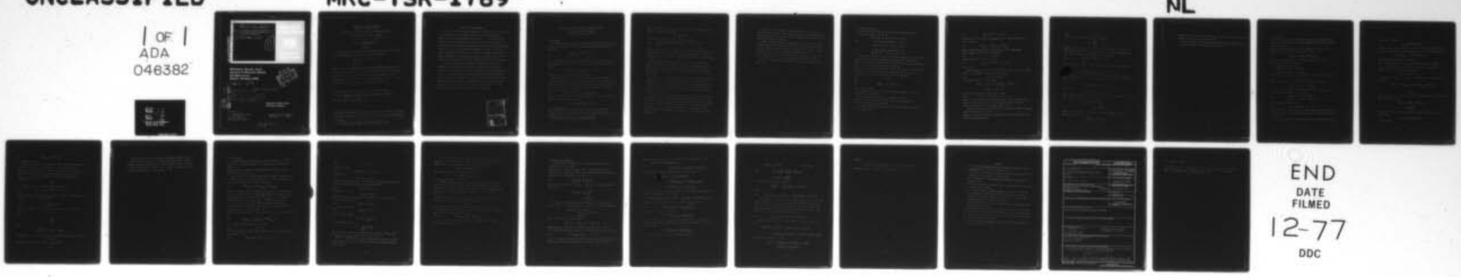

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(9) MRC Technical Summary Report #1789

(6) # A VARIATIONAL INEQUALITY APPROACH TO THE BELLMAN-DIRICHLET EQUATION FOR TWO ELLIPTIC OPERATORS.

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(11) September 1977 (12) 24p.

(Received August 24, 1977) (14) MRC-TSR-1789

(15) DAAG29-75-C-ØØ24,
NSF-MCS-77-Ø1952

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A VARIATIONAL INEQUALITY APPROACH TO THE BELLMAN-DIRICHLET EQUATION FOR TWO ELLIPTIC OPERATORS

H. Brezis*,1) and L. C. Evans**,1),2)

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ABSTRACT

We prove existence, uniqueness, and regularity properties for a solution u of the Bellman-Dirichlet equation of dynamic programming:

$$(1)\qquad \begin{cases} \max\limits_{i=1,2} \{L^i u + f^i\} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega , \end{cases}$$

where L^1 and L^2 are two second order, uniformly elliptic operators. The method of proof is to repose (1) as a variational inequality for the operator $K = L^2(L^1)^{-1}$ in $L^2(\Omega)$ and to invoke known existence theorems. For sufficiently nice f^1 and f^2 we prove in addition that $u \in H^3(\Omega) \cap C^{2,\alpha}(\Omega)$ (for some $0 < \alpha < 1$) and hence is a classical solution of (1).

AMS (MOS) Subject Classifications: 35J25, 35J60, 47H05, 49A20

Key Words: Equation of dynamic programming, Variational inequality, Regularity estimates

Work Unit Number 1 (Applied Analysis)

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Sponsored by
1) United States Army under Contract No. DAAG29-75-C-0024
2) National Science Foundation under Grant No. MCS 77-01952.

SIGNIFICANCE AND EXPLANATION

The behavior of certain dynamic systems can be modelled by an Itô stochastic differential equation (corresponding roughly to an ordinary differential equation subject to random perturbations or "noise"). A central problem in stochastic optimal control theory is to discover the best control (that is, the optimal settings of certain parameters occurring in the Itô equation describing the system) so as to maximize some performance criterion. According to certain formal reasoning, known in the literature as "Bellman's principle of dynamic programming", the problem of determining the optimal control can be converted into the problem of solving a certain nonlinear, elliptic type p.d.e.

In this paper we prove the existence of classical solutions to the Bellman p.d.e. occurring in the case that the system has only two control settings, but that the choice of control may change the "noise" affecting the system. Our treatment of this last possibility represents the main advance over previous work. We employ in our proofs the abstract theory of variational inequalities in Hilbert space: the applicability of this theory to certain other questions in control theory (to optimal stopping time probl[illegible]) has been long known, and we have now discovered a new applicatio[illegible]

A VARIATIONAL INEQUALITY APPROACH TO THE BELLMAN-DIRICHLET EQUATION FOR TWO ELLIPTIC OPERATORS

H. Brezis [*,1)] and L. C. Evans [**,1),2)]

1. Introduction.

In this paper we make use of some variational inequality techniques to prove existence, uniqueness, and regularity theorems for a solution of the Bellman-Dirichlet problem for two operators:

$$(1.1)\qquad \begin{cases} \max\limits_{i=1,2} \{L^i u + f^i\} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma . \end{cases}$$

Here Ω is a bounded domain in R^n with a smooth boundary Γ; f^1 and f^2 are given functions; and L^1 and L^2 are linear, second order, uniformly elliptic operators of the form

$$(1.2)\qquad L^i u \equiv a^i_{kj}(x)u_{x_k x_j} + b^i_k(x)u_{x_k} - \mu u \qquad (i = 1,2) .$$

(We employ the implicit summation convention throughout.)

Problem (1.1) arises as a very special case of Bellman's equation of dynamic programming for the optimal control of a certain stochastic system. More precisely, let $\omega(t)$ be a standard n-dimensional Brownian motion, and suppose that $\alpha(t)$ is a stochastic process in some appropriate class of admissible controls. We wish to study the solution $x(t)$ of the Itô differential equation

$$(1.3)\qquad dx(t) = b(x(t), \alpha(t))dt + \sigma(x(t), \alpha(t))d\omega(t) .$$

Let $x \in \Omega$ be the starting point and τ the hitting time to Γ. For a given admissible control $\alpha(\cdot)$, the payoff will be given by

$$u(x,\alpha(\cdot)) \equiv E_x[\int_0^\tau e^{-\mu t} f(x(t),\alpha(t))dt] ,$$

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where f and $\mu \geq 0$ are given (μ is the discount factor). According to certain heuristic reasoning, the optimal payoff

$$u(x) \equiv \sup_{\alpha(\cdot)} u(x,\alpha(\cdot))$$

should satisfy the Bellman-Dirichlet type differential equation:

$$(1.4) \qquad \begin{cases} \sup_{\alpha\in A} \{a_{ij}(x,\alpha)u_{x_i x_j} + b_i(x,\alpha)u_{x_i} - \mu u + f(x,\alpha)\} = 0 & \text{in } \Omega \\ \\ u = 0 & \text{on } \Gamma . \end{cases}$$

In these equations the matrix $a \equiv \frac{1}{2}\sigma^T\sigma$; and A is the set of actions, that is, the set in which the admissible controls take values. See Fleming-Rishel [3, Chapter VI] or Kushner [5, Chapter IV] for further discussion and for details of the formal derivation of (1.4).

The equations (1.1) comprise a special case of (1.4) occurring when the set of actions A consists of only two elements. Since, however, we include in (1.3) the possibility that σ depends on α, we are allowing the choice of contr l possibly to affect the "noise" in the system; and so even the simpler problem (1.1) can be rather badly nonlinear.

For the case $\Omega = R^n$, the general problem (1.4) has been studied by N. V. Krylov [4], who proved under various responsible hypotheses the existence of a unique solution $u \in W^{2,p}_{loc}(R^n)$, $(1 \leq p < \infty)$. Krylov's approach (as well as the subsequent investigation of Nisio [8]) is quite complicated and depends heavily on some delicate estimates of certain stochastic differential equations. Certain other results for the problem (1.4) have been obtained by Pucci [9].

For the simpler case of only two operators, we present in this paper a variational inequality formulation, which allows for an easy existence and uniqueness theorem. And in fact we prove that for sufficiently nice functions f^1 and f^2, (1.1) has a classical solution u, with Hölder continuous second derivatives. This is perhaps somewhat surprising, as it is well known that this much smoothness cannot be expected for the solutions of more conventional variational inequalities: see, for example, Brezis and Kinderlehrer [2].

The methods we use can be adapted to handle the case where L^1 and L^2 are parabolic operators. Also, it has been pointed out by P. L. Lions (to appear) that they lead to a solution when L^1 is elliptic and L^2 is parabolic; such a problem has been considered by Bensoussan and Lesourme.

The paper is organized this way. In Section 2 we transform (1.1) into a variational inequality and thereby, under the assumption $f^1, f^2 \in L^2(\Omega)$, find a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$. For Section 3 we assume in addition that $f^1, f^2 \in H^1(\Omega)$, and show then that $u \in H^3(\Omega)$. Classical regularity is obtained in Section 4: if $f^1, f^2 \in W^{1,p}(\Omega)$ $(p > n)$, we prove using the estimates of DeGiorgi-Moser-Stampacchia that $u \in C^{2,\alpha}(\Omega)$ for some $0 < \alpha < 1$. The Appendix comprises a proof of Lemma 2.1.

We wish to thank Ray Rishel for several conversations (with the second author) about this subject.

2. Existence and uniqueness.

Throughout the paper we will make use of these standing assumptions about the coefficients of the operators L^1 and L^2:

there exist two numbers $\Theta \geq \theta > 0$ such that

(2.1) $$\theta|\xi|^2 \leq a^i_{kj}(x)\xi_k\xi_j \leq \Theta|\xi|^2 \qquad (i = 1,2)$$

for each $x \in \Omega$ and $\xi = (\xi_1,\xi_2,\ldots,\xi_n) \in R^n$,

the coefficients are smooth functions, say

(2.2) $$a^i_{kj},\ b^i_k \in C^2(\bar{\Omega}) \qquad (k,j = 1,2,\ldots,n;\ i = 1,2)\ .$$

The main idea in the proof of Theorem 2.2 is to rewrite problem (1.1) as the variation inequality (2.10), involving a certain bounded linear operator K (defined by (2.9) below). To solve the variational inequality, we will need information about the monotonicity properties of K; and this the following lemma will provide.

Lemma 2.1. Let L^1 and L^2 be two elliptic operators of the form (1.2), the coefficients of which satisfy (2.1) and (2.2). Then there exist two constants $C_1 \geq 0$ and $C_2 > 0$, depending only on Ω and the coefficients of L^1 and L^2, such that if

(2.3) $$\mu \geq C_1\ ,$$

then

(2.4) $$\|v\|^2_{H^2(\Omega)} \leq C_2 \int_\Omega L^1v\cdot L^2v\,dx$$

for all $v \in H^2(\Omega) \cap H^1_0(\Omega)$.

The inequality is announced by Sobolevsky in [10]; for the reader's convenience we present a proof in the Appendix.

We are now ready to prove the existence and uniqueness assertion.

Theorem 2.2. Let L^1 and L^2 be two elliptic operators of the form (1.2), the coefficients of which satisfy (2.1) and (2.2).

If μ is sufficiently large, then for each $f^1, f^2 \in L^2(\Omega)$ there exists a unique $u \in H^2(\Omega) \cap H^1_0(\Omega)$ solving

$$\max_{i=1,2} \{L^i u + f^i\} = 0 \quad \text{a.e. in } \Omega \ . \tag{2.5}$$

In addition, there is a constant C, depending only on Ω and the coefficients of L^1 and L^2, such that

$$\|u\|_{H^2(\Omega)} \leq C(\|f^1\|_{L^2(\Omega)} + \|f^2\|_{L^2(\Omega)}) \ . \tag{2.6}$$

Proof. First we reduce problem (2.5) to the case where $f^1 = 0$. Indeed choose $\bar{u} \in H^2(\Omega) \cap H_0^1(\Omega)$ to solve $L^1\bar{u} = f^1$ and then set

$$v \equiv u + \bar{u} \ . \tag{2.7}$$

A simple calculation reveals that v satisfies

$$\max_{i=1,2} \{L^2 v - f, \ L^1 v\} = 0 \quad \text{a.e. in } \Omega \ , \tag{2.8}$$

for $f \equiv L^2\bar{u} - f^2$.

Define a bounded linear operator $K : L^2(\Omega) \to L^2(\Omega)$ as follows. Pick any $\phi \in L^2(\Omega)$ and let $v \in H^2(\Omega) \cap H_0^1(\Omega)$ solve $L^1 v = \phi$ in Ω; next set

$$K\phi \equiv L^2 v = L^2(L^1)^{-1}\phi \ . \tag{2.9}$$

K is monotone and coercive on $L^2(\Omega)$; indeed

$$(K\phi,\phi)_{L^2} = \int_\Omega L^2 v \cdot L^1 v \, dx \geq \frac{1}{C_2} \|v\|^2_{H^2(\Omega)} \quad \text{by Lemma 2.1}$$

$$\geq c\|\phi\|^2_{L^2} \quad \text{for some } c > 0 \ .$$

In terms of the new unknown ϕ, problem (2.5) now reads

$$\phi \leq 0, \ K\phi - f \leq 0, \ \phi(K\phi - f) = 0 \quad \text{a.e. in } \Omega \ ; \tag{2.10}$$

that is, ϕ is the solution of a variational inequality in $L^2(\Omega)$ associated with the operator K and the convex set $\{\psi \leq 0\}$. Existence and uniqueness of a solution follow from standard results (see, for example Stampacchia [11]).

Estimate (2.6) follows from (2.8), (2.9) (with $\psi = 0$), and the standard L^2 *a priori* estimates for the operators L^1 and L^2. ■

Remark 2.3. For the purpose of proving regularity results it is also convenient to view (2.10) as the multivalued equation

$$(2.11) \qquad K(\phi) + \beta(\phi) \ni f ,$$

where β is the maximal monotone graph on R defined by

$$\beta(x) = \begin{cases} 0 & x < 0 \\ [0,\infty) & x = 0 \\ \phi & x > 0 . \end{cases} \qquad \blacksquare$$

Remark 2.4. Inequality (2.4) can be used to prove an existence and regularity theorem for solutions of certain second order, uniformly elliptic equations in nondivergence form, the coefficients of which are discontinuous. Consider the problem

$$(2.12) \qquad \begin{cases} a_{kj}u_{x_k x_j} + b_k u_{x_k} - \mu u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma . \end{cases}$$

Suppose that the (possibly discontinuous) coefficients a_{ij} can be decomposed into the form

$$(2.13) \qquad a_{kj} = a^1_{kj}\chi_E + a^2_{kj}\chi_{\Omega\setminus E} \qquad (i,j = 1,2,\ldots,n) ,$$

where the a^i_{kj}, $(i = 1,2)$ are smooth functions and χ_E is the characteristic function of some measurable subset $E \subset \Omega$. Let us derive an a priori H^2 estimate for a solution of (2.12). Set

$$L^i u \equiv a^i_{kj}u_{x_k x_j} + b_k u_{x_k} - \mu u \qquad (i = 1,2) ;$$

then for a.e. x

$$(L^1u(x) - f(x))(L^2u(x) - f(x)) = 0 ,$$

since, by (2.12), at least one of the two terms is zero. We integrate this identity over Ω and make some simple estimates to obtain

$$\int_\Omega L^1u \cdot L^2u \, dx \le \epsilon \|u\|^2_{H^2(\Omega)} + \left(\frac{C}{\epsilon} + 1\right) \|f\|^2_{L^2(\Omega)} .$$

By Lemma 2.1 the left hand side is greater than or equal to $\frac{1}{C_2}\|u\|^2_{H(\Omega)}$ (if μ is sufficiently large); thus

$$\|u\|_{H^2(\Omega)} \le C\|f\|_{L^2(\Omega)} .$$

An existence proof for a solution of (2.12) can now be carried out, based on this *a priori* inequality and the continuation of parameter method.

Notice that problem (1.1) has the same form as (2.12)-(2.13), except that the set E is not known beforehand, and in fact depends upon the solution u. ■

3. H^3 estimates.

In this and the next section we shall further exploit our variational inequality reformulation of (1.1) to derive, under stronger assumptions on f^1 and f^2, some more regularity properties for u.

Theorem 3.1. Suppose, in addition to the other hypotheses of Theorem 2.2, that $f^1, f^2 \in H^1(\Omega)$. Then $u \in H^3(\Omega)$; and we have the estimate

$$\|u\|_{H^3(\Omega)} \leq C(\|f^1\|_{H^1(\Omega)} + \|f^2\|_{H^1(\Omega)}) , \tag{3.1}$$

the constant C depending only on Ω and the coefficients of L^1 and L^2.

Proof. The proof consists of three parts:

A. Approximation of problem (2.11)

B. Derivation of interior H^3 estimates

C. Derivation of boundary H^3 estimates.

A. Approximation. According to (2.11), the function v defined by (2.7) satisfies

$$L^2 v + \beta(L^1 v) \ni f . \tag{3.2}$$

Since $f^1, f^2 \in H^1(\Omega)$, we have $f \in H^1(\Omega)(f \equiv L^2\bar{u} - f^2)$, with the estimate

$$\|f\|_{H^1(\Omega)} \leq C(\|f^1\|_{H^1(\Omega)} + \|f^2\|_{H^1(\Omega)}) . \tag{3.3}$$

To avoid any calculations involving multivalued operators, let us now select a sequence β_k of smooth functions on R such that

$$\begin{cases} \beta_k(x) = 0 \quad \text{for} \quad x \leq 0; \quad \beta_k'(x) \geq 0 ; \\ \beta_k(x) \to \infty \quad \text{as} \quad k \to \infty, \quad \text{for} \quad x > 0 . \end{cases} \tag{3.4}$$

We consider the approximate problems

$$L^2 v_k + \beta_k(L^1 v_k) = f \qquad (k = 1,2,\ldots) , \tag{3.5}$$

the unique solvability of which in the class $H^2(\Omega) \cap H^1_0(\Omega)$ follows from rewriting (3.5) as $K(\phi) + \beta_k(\phi) = f$ and invoking standard monotone operator theory (see Brezis [1] for example).

For simplicity of notation let us drop the subscript k until the end of the proof.

B. Interior H^3 estimates. Let

$$g_{(h)}(x) \equiv \frac{g(x + he_i) - g(x)}{h}$$

denote the difference quotient of a function g in some fixed i^{th} coordinate direction $e_i (1 \le i \le n)$. We now choose a $C_0^\infty(\Omega)$ cutoff function ζ, multiply equation (3.5) by

$$(\zeta^2 (L^1 v)_{(h)})_{(-h)} \qquad 0 < h < \mathrm{dist}(\mathrm{supt}\ \zeta, \Gamma) ,$$

and then change variables to obtain the expression

$$(3.6) \qquad \int_\Omega (L^2 v)_{(h)} (L^1 v)_{(h)} \zeta^2 dx + \int_\Omega (\beta(L^1 v))_{(h)} (L^1 v)_{(h)} \zeta^2 dx = \int_\Omega f_{(h)} (L^1 v)_{(h)} \zeta^2 dx .$$

Since β is nondecreasing, the second term on the left hand side is nonnegative. Furthermore, notice that the L^2 norm of the difference

$$(L^i v)_{(h)} \zeta - L^i (v_{(h)} \zeta) \qquad (i = 1,2)$$

can be dominated by $C\|v\|_{H^2(\Omega)}$. Using these two observations, we now derive from (3.6) the estimate

$$\int_\Omega L^2 \bar{v} \cdot L^1 \bar{v}\, dx \le \epsilon \|\bar{v}\|^2_{H^2(\Omega)} + \left(C + \frac{C}{\epsilon}\right) (\|v\|^2_{H^2(\Omega)} + \|f\|^2_{H^1(\Omega)})$$

for

$$\bar{v} \equiv v_{(h)} \zeta .$$

Since $\bar{v} \in H^2(\Omega) \cap H_0^1(\Omega)$, we can make use of Lemma 2.1, and then choose $\epsilon > 0$ sufficiently small, to obtain the inequality

$$\|\bar{v}\|^2_{H^2(\Omega)} \le C (\|v\|^2_{H^2(\Omega)} + \|f\|^2_{H^1(\Omega)})$$

$$\le C\|f\|^2_{H^1(\Omega)} .$$

In this expression the constant C depends on ζ and other known quantities, but not on h. Thus

$$v \in H^3(\Omega') \text{ for each } \Omega' \subset\subset \Omega ,$$

with the estimate

(3.7) $$\|v\|_{H^3(\Omega')} \le C(\Omega')\|f\|_{H^1(\Omega)} .$$

C. Boundary H^3 estimates.

Consider now some portion Γ' of the boundary, which - upon a smooth local change of coordinates if necessary - we may assume to lie in the plane $x_n = 0$, with $B_R^+ \equiv \{|x| < R, x_n > 0\} \subset \Omega \subset \{x_n > 0\}$, for some $R > 0$. First of all we consider the tangential difference quotients $v_{(h)}$ in the directions $e_i (1 \le i \le n-1)$; since $v \equiv 0$ on Γ', $v_{(h)} \equiv 0$ there also. Therefore if we choose a smooth cutoff function ζ vanishing near $|x| = R$, then

$$\bar{v} = v_{(h)}\zeta$$

is in $H_0^1(B_R^+)$; and so the considerations from before imply that

(3.8) $$v_{x_1}, v_{x_2}, \ldots, v_{x_{n-1}} \in H^2(B_{R'}^+) \qquad (0 < R' < R) .$$

Thus it remains only to show that $v_{x_n x_n x_n}$ belongs to $L^2(B_{R'}^+)$; all the other third derivatives of v are in $L^2(B_{R'}^+)$ by (3.8).

Set

(3.9) $$w \equiv L^1 v = a_{nn}^1(x) v_{x_n x_n} + (L^1 v)' ,$$

with $(L^1 v)' \in H^1(B_{R'}^+)$ by (3.8). In terms of the new function w, (3.5) now reads

(3.10) $$\lambda(x) w + \beta(w) = g$$

for

$$\lambda \equiv \frac{a_{nn}^2}{a_{nn}^1}$$

and

$$g \equiv \frac{a_{nn}^2}{a_{nn}^1}(L^1 v)' - (L^2 v)' + f \in H^1(B_{R'}^+) .$$

By hypotheses (2.1) and (2.2), λ is a positive, bounded, and smooth function. Therefore

$$w = (\lambda(x) + \beta)^{-1} g$$

belongs also to $H^1(B_{R'}^+)$ and so by (3.11), $v_{x_n x_n} \in H^1(B_{R'}^+)$.

The calculations in parts B and C preceding provide bounds, independent of k, for the $H^3(\Omega)$ norm of the solutions v_k of the approximate problems (3.5). Standard convergence arguments now imply that the v_k converge weakly in $H^3(\Omega)$ (and strongly in $H^2(\Omega)$) to the solution v of (3.2). Hence $v \in H^3(\Omega)$, and estimate (3.1) follows easily from the estimates on v, (3.3), and (2.7). ■

4. $C^{2,\alpha}$ estimates.

We now prove that under somewhat stronger regularity assumptions on f^1 and f^2, problem (1.1) has a classical solution u, with locally Hölder continuous second derivatives in Ω.

Theorem 4.1. Assume, in addition to the other hypotheses of Theorem 2.2, that $f^1, f^2 \in W^{1,p}(\Omega)$ for some $n < p < \infty$.

Then the solution u of (1.1) has locally Hölder continuous second derivatives in Ω; and for each $\Omega' \subset\subset \Omega$, there are constants $0 < \alpha < 1$ and C, depending only on p, Ω', and the coefficients of L^1 and L^2, such that

$$\|u\|_{C^{2,\alpha}(\bar{\Omega}')} \leq C(\|f^1\|_{W^{1,p}(\Omega)} + \|f^2\|_{W^{1,p}(\Omega)}) \tag{4.1}$$

Remark 4.2. In the case that $f^1, f^2 \in W^{1,p}(\Omega)$ for some $2 \leq p < n$, then the proof below is easily modified to show $u \in W^{2,p^*}_{loc}(\Omega)$ $\left(\frac{1}{p^*} \equiv \frac{1}{p} - \frac{1}{n}\right)$. It would be of interest to discover whether the weaker assumption $f^1, f^2 \in L^p(\Omega)$ $(2 \leq p < \infty)$ implies that $u \in W^{2,p}(\Omega)$ (or that $f^1, f^2 \in L^\infty(\Omega)$ implies $u \in W^{2,p}(\Omega)$ for some $p > n$.)

We do not know, even for $f^1, f^2 \in C^\infty(\Omega)$, whether the solution belongs to $W^{3,p}(\Omega)$ for every p or to $C^{2,\alpha}(\Omega)$ for all $0 < \alpha \leq 1$. It is easy to construct examples for which $u \in C^{2,1}(\Omega)$, $u \notin C^3(\Omega)$. ■

Proof. As in the proof of Theorem 2.2 we reduce (1.1) to the form (2.8), with $f \in W^{1,p}$ and

$$\|f\|_{W^{1,p}(\Omega)} \leq C(\|f^1\|_{W^{1,p}(\Omega)} + \|f^2\|_{W^{1,p}(\Omega)}) \tag{4.2}$$

Set $\varepsilon = \frac{\theta}{2\Theta}$ (θ and Θ from hypothesis (2.1)). Then

$$M \equiv L^2 - \varepsilon L^1 \tag{4.3}$$

is a uniformly elliptic second order operator with smooth top order coefficients $\bar{a}_{k\ell}$. We may rewrite (2.8) as

$$\max\{Mv + \varepsilon L^1 v - f, L^1 v\} = 0 \text{ a.e. in } \Omega \tag{4.4}$$

Set

(4.5) $$w \equiv Mv - f ;$$

and then (4.4) becomes

(4.6) $$\max\{w + \varepsilon L^1 v,\ L^1 v\} = 0 \quad \text{a.e. in } \Omega ,$$

or equivalently

(4.7) $$\varepsilon L^1 v + w^+ = 0 \quad \text{a.e. in } \Omega .$$

It follows from (4.5) and Theorem 3.1 that $w \in H^1(\Omega)$. Let us apply the operator M to (4.7); the resulting expression makes sense in the space $H^{-1}(\Omega)$:

(4.8) $$\varepsilon M(L^1 v) + Mw^+ = 0 .$$

The commutator

(4.9) $$Dv \equiv M(L^1 v) - L^1(Mv)$$

contains terms involving at most <u>third</u> order derivatives of v. We substitute (4.9) and (4.5) into (4.8) to obtain

(4.10) $$\varepsilon[L^1(w+f) + Dv] + Mw^+ = 0 .$$

Let us rewrite (4.10) as

(4.11) $$(\tilde{a}_{k\ell} w_{x_k})_{x_\ell} = R_1 + R_2 ,$$

where $\tilde{a}_{k\ell} \equiv \varepsilon a^1_{k\ell} + \bar{a}_{k\ell}\chi_{\{w>0\}}$,

$$R_1 \equiv -\varepsilon L^1 f = \sum_{|\alpha|\le 1} D^\alpha(g_\alpha)$$

for $g_\alpha \in L^p(\Omega)$ $(p > n)$, and

$$R_2 \equiv \sum_{\substack{|\alpha|\le 1\\ |\beta|\le 2}} D^\alpha(a_{\alpha\beta} D^\beta v)$$

where $a_{\alpha\beta} \in L^\infty(\Omega)$. (Note that for example a term like $b(x)(w^+)_{x_k} = (bw^+)_{x_k} - b_{x_k} w^+ = (b\chi_{[w>0]} w)_{x_k} - b_{x_k}\chi_{[w>0]} w$ can be included in R_2 by (4.5)).

Since $\varepsilon > 0$ the coefficients $\tilde{a}_{k\ell}$ are bounded measurable and satisfy a uniform ellipticity condition; the interior estimates of DeGiorgi-Moser-Stampacchia therefore apply to (4.11).

Since $v \in H^3(\Omega)$ by Theorem 3.1 it follows that $v_{x_i x_j} \in L^{2^*}(\Omega)$ $(i,j = 1,\ldots,n)$, where $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$ (with the usual modifications when $n = 1$ or 2). We may therefore apply Theorem 5.4 in Stampacchia [11] to equation (4.11) to conclude that

$$w \in L^{2^{**}}(\Omega_1) \quad \text{for any domain } \Omega_1 \subset\subset \Omega .$$

Then by equation (4.5) and the standard L^p estimates,

$$v \in W^{2,2^{**}}(\Omega_2) \qquad \Omega_2 \subset\subset \Omega_1 ,$$

and so $v_{x_i x_j} \in L^{2^{**}}(\Omega_2)$ $(i,j = 1,\ldots,n)$.

Continuing this bootstrap procedure $k = [n/2]$ times, we arrive finally at a subdomain $\Omega' \subset\subset \Omega$ on which w satisfies equation (4.11), the right hand side of which can be expressed as $(\tilde{g}_j)_{x_j}$, $\tilde{g}_j \in L^p(\Omega')$, $p > n$. As such Theorem 6.2 in Stampacchia [11] implies that w is Hölder continuous with some exponent $0 < \lambda < 1$ in any domain $\Omega'' \subset\subset \Omega'$.

Now since $f \in W^{1,p}(\Omega)$ for $p > n$, $f \in C^{\gamma}(\bar{\Omega})$, with $\gamma = 1 - \frac{n}{p}$. Thus in the domain Ω'', v solves according to (4.5) the elliptic equation

$$Mv = f + w ,$$

the right hand side of which is Hölder continuous with exponent $\alpha = \min(\gamma,\lambda)$.

By the standard interior Schauder estimates therefore, $v \in C^{2,\alpha}(\Omega'')$ for $\Omega''' \subset\subset \Omega''$; and the argument above also provides the estimate

$$\|v\|_{C^{2,\alpha}(\Omega''')} \leq C(\Omega''')\|f\|_{W^{1,p}(\Omega)} . \tag{4.12}$$

Since the solution u of (1.1) is related to v by (2.7), where $\bar{u} \in C^{2,\gamma}(\bar{\Omega})$, inequality (4.1) is proved. ■

5. Appendix: Proof of Lemma 2.1.

The following proof of inequality (2.4) is based on Ladyženskaja [6, §3] and Ladyženskaja-Ural'ceva [7, p. 182].

Lemma 5.1. Let $A = ((a_{ij}))$, $B = ((b_{ij}))$, and $C = ((c_{ij}))$ be three real $n \times n$ symmetric matrices. Suppose that A and B are nonnegative definite, each with smallest eigenvalue greater than or equal to $\nu \geq 0$. Then

$$a_{ij}c_{ik}b_{kl}c_{jl} \geq \nu^2 \sum_{i,j=1}^{n} c_{ij}^2 . \tag{5.1}$$

Proof. Consider first the case $A = \mathrm{dia}(a_{11},\ldots,a_{nn})$, $a_{ii} \geq \nu$; then the left hand side of (3.1) reads

$$a_{ii}c_{ik}b_{kl}c_{il} \geq a_{ii}\nu \sum_{j=1}^{n} c_{ij}^2$$

$$\geq \nu^2 \sum_{i,j=1}^{n} c_{ij}^2 .$$

In the general case we note that the left hand side of (5.1) is the trace of the matrix $D \equiv ACBC$. Choose an orthogonal matrix E such that $A' \equiv EAE^{-1}$ is diagonal. Then

$$\begin{aligned} a_{ij}c_{ik}b_{kl}c_{jl} &= \mathrm{tr}(D) = \mathrm{tr}(EDE^{-1}) \\ &= \mathrm{tr}(EAE^{-1}\cdot ECE^{-1}\cdot EBE^{-1}\cdot ECE^{-1}) \\ &= \mathrm{tr}(A'C'B'C') , \end{aligned}$$

where $B' \equiv EBE^{-1}$, $C' \equiv ECE^{-1}$. By the first case

$$\mathrm{tr}(A'C'B'C') \geq \nu^2 \sum_{i,j=1}^{n} (c'_{ij})^2 = \nu^2 \sum_{i,j=1}^{n} c_{ij}^2 . \qquad \blacksquare$$

Proof of Lemma 2.1. We may assume, with no loss of generality, that $u \in C^3(\bar{\Omega})$, $u = 0$ on Γ. We investigate first the case that L^1 and L^2 have no lowest order term, that is,

$$L^i u = a^i_{kj}u_{x_k x_j} + b^i_k u_{x_k} \qquad (i = 1,2) . \tag{5.2}$$

Let $\Gamma' \subset \Gamma$ be some given portion of the boundary, which-upon a change of coordinates if necessary-we may assume to lie in the plane $x_n = 0$ (with $\Omega \subset \{x_n > 0\}$). Choose a smooth cutoff function ζ, $0 \le \zeta \le 1$, such that $\zeta(x) = 0$ for x near $\Gamma \setminus \Gamma'$. Then

$$\int_\Omega \zeta L^1 u \cdot L^2 u \, dx = \int_\Omega \zeta a^1_{ij} u_{x_i x_j} a^2_{kl} u_{x_k x_l} \, dx \tag{5.3}$$

$$+ \int_\Omega \zeta (a^1_{ij} u_{x_i x_j} b^2_k u_{x_k} + a^2_{kl} u_{x_k x_l} b^1_i u_{x_i} + b^1_i u_{x_i} b^2_k u_{x_k}) dx \, .$$

We transform the first term on the right by integrating by parts twice:

$$\int_\Omega \zeta a^1_{ij} u_{x_i x_j} a^2_{kl} u_{x_k x_l} \, dx = \int_\Omega \zeta a^1_{ij} u_{x_i x_k} a^2_{kl} u_{x_j x_l} \, dx$$

$$+ \int_\Omega (\zeta a^1_{ij} a^2_{kl})_{x_j} u_{x_i x_k} u_{x_l} - (\zeta a^1_{ij} a^2_{kl})_{x_k} u_{x_i x_j} u_{x_l} \, dx$$

$$+ \int_\Gamma \zeta a^1_{ij} a^2_{kl} (u_{x_i x_j} u_{x_l} n_k - u_{x_i x_k} u_{x_l} n_j) ds \, ;$$

here $n = (n_1, \ldots, n_n)$ is the outward unit normal. Call the integrand of the last term I. Then the preceding calculation and (5.3) imply

$$\int_\Omega \zeta a^1_{ij} u_{x_i x_k} a^2_{kl} u_{x_j x_l} \, dx \le \int_\Omega \zeta L^1 u \cdot L^2 u \, dx + \epsilon \sum_{i,j=1}^{n} \int_\Omega u^2_{x_i x_j} \, dx + \frac{C}{\epsilon} \int_\Omega |\nabla u|^2 dx - \int_\Gamma I \, ds \, . \tag{5.4}$$

By Lemma 5.1, with $A = ((a^1_{ij}))$, $B = ((a^2_{kl}))$, and $C = ((u_{x_i x_j}))$, we have

$$\nu^2 \sum_{i,j=1}^{n} \int_\Omega \zeta u^2_{x_i x_j} \, dx \le \int_\Omega \zeta a^1_{ij} u_{x_i x_k} a^2_{kl} u_{x_j x_l} \, dx \, . \tag{5.5}$$

Furthermore, since $u \equiv 0$ on Γ, we have

$$I(x) = \zeta [a^1_{ij} a^2_{nn} u_{x_i x_j} u_{x_n} - a^1_{in} a^2_{kn} u_{x_i x_k} u_{x_n}]$$

for $x \in \Gamma'$. When $1 \le i,j \le n-1$, $u_{x_i x_j} = 0$. In addition, for $i = j = k = n$ the two terms involving only normal differentiations cancel; this also happens for $j = k = n, i$ arbitrary. Thus

$$I(x) = \sum_{j=1}^{n-1} \zeta [a^1_{nj} a^2_{nn} - a^1_{nn} a^2_{jn}] \frac{1}{2} \frac{d}{dx_j} (u^2_{x_n})$$

for $x \in \Gamma'$. Since $\zeta = 0$ on $\Gamma \setminus \Gamma'$ we may therefore calculate

$$\left|\int_\Gamma I(x)ds\right| = \left|\int_{\Gamma'} I(x)dx'\right| \qquad x' = (x_1,\dots,x_{n-1})$$

(5.6)
$$= \left|\frac{1}{2}\int_{\Gamma'} u_{x_n}^2 \sum_{j=1}^{n-1} (\zeta[a_{nj}^1 a_{nn}^2 - a_{nn}^1 a_{jn}^2])_{x_j} dx'\right|$$

$$\leq C\int_{\Gamma'} u_{x_n}^2 dx' \leq C\int_\Gamma \left(\frac{\partial u}{\partial n}\right)^2 ds .$$

In view of (5.4)-(5.6) and the trace inequality

$$\int_\Gamma \left(\frac{\partial u}{\partial n}\right)^2 ds \leq \epsilon \sum_{i,j=1}^n \int_\Omega u_{x_j x_j}^2 dx + C(\epsilon)\int_\Omega |\nabla u|^2 dx ,$$

we have

(5.7)
$$\nu^2 \sum_{i,j=1}^n \int_\Omega \zeta u_{x_i x_j}^2 dx \leq \int_\Omega \zeta L^1 u\cdot L^2 u\, dx + \epsilon \sum_{i,j=1}^n \int_\Omega u_{x_i x_j}^2 dx + C(\epsilon)\int_\Omega |\nabla u|^2 dx .$$

Next we decompose Γ into the union of finitely many pieces Γ_i' $(i = 1,\dots,k)$, each of which can be mapped as above by a smooth change of coordinates into the plane $x_n = 0$. Let ζ_i $(i = 1,\dots,k)$ be a smooth partition of unity on Ω, with $\zeta_i \equiv 0$ near $\Gamma\backslash\Gamma_i'$. We sum the finite number of inequalities (5.7) resulting from these choices of ζ, and then select $\epsilon > 0$ sufficiently small to derive the estimate

(5.8)
$$\sum_{i,j=1}^n \int_\Omega u_{x_i x_j}^2 dx \leq C\int_\Omega L^1 u\cdot L^2 u\, dx + C\int_\Omega |\nabla u|^2 dx .$$

Finally, we assume the operators L^1 have the form (1.2), and set $M^i u \equiv L^i u + \mu u$. Then

$$\int_\Omega L^1 u\cdot L^2 u\, dx = \int_\Omega M^1 u\cdot M^2 u\, dx - \mu\int_\Omega (u\cdot M^1 u + u\cdot M^2 u)dx + \mu^2\int_\Omega u^2 dx$$

$$\geq C\sum_{i=1}^n \int_\Omega u_{x_i x_j}^2 dx - C\int_\Omega |\nabla u|^2 dx - \mu\int_\Omega (u\cdot M^1 u + u\cdot M^2 u)dx + \mu^2\int_\Omega u^2 dx$$

by (5.8) (with M^i replacing L^i). Now

$$-\mu\int_\Omega u\cdot M^i u = \mu\int_\Omega a_{kl}^i u_{x_k} u_{x_l} dx + \mu\int_\Omega uu_{x_l}(a_{kl}^i)_{x_k} - b_k^i uu_{x_k} dx$$

$$\geq \mu\nu\int_\Omega |\nabla u|^2 dx - \mu C\int_\Omega |u||\nabla u| dx ;$$

therefore

$$\int_\Omega L^1u\cdot L^2u\,dx \geq C\sum_{i,j=1}^{n}\int_\Omega u^2_{x_ix_j}dx + (\mu^2 - C\mu)\int_\Omega u^2 dx + \frac{\mu\nu}{2}\int_\Omega |\nabla u|^2 dx .$$

We complete the proof by choosing $\mu > 0$ sufficiently large. ∎

REFERENCES

1. H. Brezis, Operateurs maximaux monotones et semigroupes de contractions dans les espaces Hilbert, Math. Studies 5, North Holland, 1973.

2. H. Brezis and D. Kinderlehrer, The smoothness of solutions of nonlinear variational inequalities, Indiana U. Math. J. 23 (1974), 831-844.

3. W. H. Fleming and R. W. Rishel, Deterministic and Stochastic Optimal Control, Springer, NY 1975.

4. N. V. Krylov, Control of a solution of a stochastic integral equation, Theory Prob. and Appl. 17 (1972), 114-131.

5. H. J. Kushner, Stochastic Stability and Control, Academic Press, New York, 1967.

6. O. A. Ladyženskaja, On integral inequalities, the convergence of approximative methods, and the solution in functional for linear elliptic operators, Vestnik, Leningrad State University 7 (1958), 60-69 (Russian).

7. O. A. Ladyženskaja and N. N. Ural'ceva, Linear and Quasilinear Elliptic Equations, Academic Press, New York, 1968.

8. M. Nisio, Remarks on stochastic optimal controls, Japan J. Math. 1 (1975), 159-183.

9. C. Pucci, Operatori ellittici estremanti, Ann. Mat. Pura. Appl. 72 (1966), 141-170.

10. P. Sobolevsky, Equations with operators constituting an acute angle, Doklady. Akad. Nauk. USSR 11b(1957), 754-757 (Russian).

11. G. Stampacchia, Équations elliptiques du second ordre à coefficients discontinus, University of Montreal Press, 1966.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 1789	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A VARIATIONAL INEQUALITY APPROACH TO THE BELLMAN-DIRICHLET EQUATION FOR TWO ELLIPTIC OPERATORS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) H. Brezis and L. C. Evans		8. CONTRACT OR GRANT NUMBER(s) DAAG29-75-C-0024 MCS 77-01952
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 1 (Applied Analysis)
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below.		12. REPORT DATE September 1977 13. NUMBER OF PAGES 19
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Equation of dynamic programming
Variational inequality
Regularity estimates

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We prove existence, uniqueness, and regularity properties for a solution u of the Bellman-Dirichlet equation of dynamic programming:

$$(1)\quad \begin{cases} \max\limits_{i=1,2} \{L^i u + f^i\} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega , \end{cases}$$

where L^1 and L^2 are two second order, uniformly elliptic operators. The method of proof is to repose (1) as a variational inequality for the operator

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20. ABSTRACT - Cont'd.

$K = L^2(L^1)^{-1}$ in $L^2(\Omega)$ and to invoke known existence theorems. For sufficiently nice f^1 and f^2 we prove in addition that $u \in H^3(\Omega) \cap C^{2,\alpha}(\Omega)$ (for some $0 < \alpha < 1$) and hence is a classical solution of (1).